PLISKA

STUDIA MATHEMATICA
BULGARICA

ПЛИСКА

БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

# A THEOREM OF WHITNEY'S TYPE IN $R^2$

BLAGOVEST H. SENDOV, MILKO D. TAKEV

In [1, 2] H. Whitney proved the following classical theorem in approximation theory and numerical analysis:

Theorem 1. (H. Whitney) *For each integer $n \geq 1$, there is a number $W_n$ with the following property. For any interval $[a, b]$ and any bounded function $f$ on $[a, b]$ there is a polynomial $P$ of a degree at most $n-1$ such that*

$$(0.1)\qquad |f(x)-P(x)| \leq W_n \omega_n(f; (b-a)/n); \quad x \in [a, b].$$

Here $\omega_n(f, \delta)$ denotes the modulus of continuity:

$$\omega_n(f; \delta) := \sup_{|h| \leq \delta} \{|\Delta_h^n f(t)| : t, t+nh \in [a, b]\};$$

$$\Delta_h^n f(x) := \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} f(x+ih).$$

K. Ivanov [3] proved Theorem 1 for functions $f$ integrable in the Lebesgue sense on $[a, b]$.

For a finite interval $I$, the smallest possible constant $W_n$ in (0.1) is clearly independent of the length of $I$. Therefore, there are essentially three distinct cases to consider: $I=[0, 1]$, $I^*=[0, \infty)$ and $I^{**}=(-\infty, \infty)$. The smallest possible constants $W_n$, $W_n^*$ and $W_n^{**}$ in each of these cases are called the Whitney constants.

H. Whitney [1] gave good estimates of the constants $W_n^*$ and $W_n^{**}$. It is proved in [4, 5] that the Whitney constants $W_n$ are bounded by a number which does not depend on $n$.

Theorem 2. (Improved Theorem of H. Whitney [5]) *For any function $f$, defined, bounded and integrable on $[0, 1]$ and for each integer $n \geq 1$, there is a polynomial $P$ of a degree at most $n-1$ such that*

$$(0.2)\qquad |f(x)-P(x)| \leq 6\,\omega_n(f; 1/(n+1)); \quad x \in [0, 1].$$

J. Brudnyi [7] proved an analogy of Theorem 1 for approximation of functions of several variables by quasipolynomials.

The purpose of this paper is to improve the J. Brudnyi's theorem in the sense of Theorem 2.

**1. Definitions and denotations.** We shall fix integers $n \geq 1$, $m \geq 1$ and will introduce the following notations: $\square$ stands for the set

$$(1.1)\qquad \square := \{(x, y) \in R^2 : 0 \leq x \leq n+1,\ 0 \leq y \leq m+1\}.$$

Let $L(M)$ be the set of defined, bounded and integrable in the Lebesgue sense functions on the set $M$ (where $M$ is a measurable subset of $R^2$ or $R^1$), equipped with the uniform norm $\|\cdot\|$.

*PLISKA Studia mathematica bulgarica, Vol. 11, 1991, p. 78-85.*

Let $Q_{n,m}(\square)$ denote the set of quasipolynomials of the $x$-th power not exceeding $n$ and of the $y$-th power not exceeding $m$:

(1.2) $$Q_{n,m}(\square) := P_{n,m}(x, y) = \sum_{i=0}^{m} f_i(x)y^i + \sum_{j=0}^{n} g_j(y)x^j,$$

where $f_i(x) \in L([0, n+1])$ for $i=0, 1, \ldots, m$; $g_j(y) \in L([0, m+1])$ for $j=0, 1, \ldots, n$.

The best uniform approximation of the function $f(x, y) \in L(\square)$ by means of elements of $Q_{n,m}(\square)$ is

(1.3) $$E(Q_{n,m}(\square); f) = E(f) := \inf\{\|f-P\|: P \in Q_{n,m}(\square)\}.$$

As a characteristic of $E(f)$ we shall use the following modulus of continuity

(1.4) $$\omega_{n,m}(f; \square) = \omega_{n,m}(f; 1, 1) = \sup_{0\leq\zeta,\ \eta\leq 1} \{|\Delta_{\zeta,\eta}^{n,m} f(x, y)|:$$

$$(x, y), (x+n\zeta, y+m\eta) \in \square;\ \Delta_{\zeta,\eta}^{m,n} f(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{n+m+i+j} \binom{n}{i}\binom{m}{j}$$

$$\times f(x+i\zeta, y+j\eta)\}.$$

Let $x=\mu+\sigma$, $\mu=0, 1, \ldots, n$, $0\leq\sigma\leq 1$, $y=\nu+\tau$, $\nu=0, 1, \ldots, m$, $0\leq\tau\leq 1$;

(1.5) $l_{n,k}(t) = \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{t-j}{k-j}$ for $k=0, 1, \ldots, n$, are the basic Lagrange polynomials for the knots $0, 1, \ldots, n$.

The following operator was introduced in [5] for all functions $f \in L([0, n+1])$:

(1.6) $$\varphi_n(f; x) = \varphi_n(f; \mu+\sigma) = \frac{(-1)^{n+\mu}}{\binom{n}{\mu}} \int_0^1 \Delta_u^n f(x-\mu u)\, du.$$

By analogy with (1.6), we define in $L(\square)$ the operator $\varphi_{n,m}(f; x, y)$ that will play a very important role in our further work

(1.7) $$\varphi_{n,m}(f; x, y) = \varphi_{n,m}(f; \mu+\sigma, \nu+\tau)$$

$$= \frac{(-1)^{n+m+\mu+\nu}}{\binom{n}{\mu}\binom{m}{\nu}} \int_0^1 \int_0^1 \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{n+m+i+j} \binom{n}{i}\binom{m}{j} f(\mu+\sigma+(i-\mu)n, \nu+\tau$$

$$+(j-\nu)r)\, du\, dr.$$

Obviously, the inequalities

(1.8) $$|\varphi_{n,m}(f; x, y)| \leq \omega_{n,m}(f; \square)/\binom{n}{\mu}\binom{m}{\nu}$$

hold for $x \in [\mu, \mu+1]$; $y \in [\nu, \nu+1]$ and $\mu=0, 1, \ldots, n$, $\nu=0, 1, \ldots, m$.

**2. Preliminaries.** We shall use the following lemmas from [5]:

Lemma 1. *For each natural number $n$,*

$$\max\{\sum_{=0}^{n} |l_{n,j}(x)|/\binom{n}{j}: 0\leq x\leq 1\} = 1,$$

*where* $l_{n,j}(x)$ *are as in* (1.5).

Lemma 2. *If*

(2.1) $$\mu_{n,\nu}=\sum_{j=0}^{n}\frac{1}{\binom{n}{j}}\max\{|l_{n,j}(x)|:\ \nu\leq x\leq\nu+1\},$$

*then* $\mu_{n,\nu}\leq\frac{1+\sigma_\nu+\sigma_{\nu+1}}{\binom{n}{\nu}}$, *where* $\nu=0,1,\ldots,[(n-1)/2]$, $\sigma_\gamma=1+1/2+\cdots+1/\nu$; $\sigma_0=0$, *and (in particular)*

(2.2) $$\mu_{n,0}=1+\sum_{j=1}^{n}\frac{1}{\binom{n}{j}}\max\{|l_{n,j}(x)|:\ 0\leq x\leq 1\}\leq 2.$$

Lemma 3. *If* $f\in L([0,n+1])$, *then we have the following representation of* $f$

(2.3) $$\int_0^{\mu+\sigma}f(n)dn=\sum_{p=0}^{n}l_{n,p}(\mu+\sigma)\int_0^{p}f(n)dn+\sum_{p=0}^{n}\int_0^{\sigma}\varphi_n(f;p+n)\,l_{n,p}(\mu+\sigma-n)dn,$$

*where* $\mu=0,1,\ldots,n$, $0\leq\sigma\leq 1$.

Now we are ready to formulate and prove an analogous formula to (2.3).

Theorem 3. If $f\in L(\square)$, *then the following equation is true*

(2.4) $$\int_0^{\mu+\sigma}\int_0^{\nu+\tau}f(n,r)dndr=\sum_{p=0}^{n}\sum_{q=0}^{m}l_{n,p}(\mu+\sigma)l_{m,q}(\nu+\tau)\int_0^{p}\int_0^{q}f(n,r)dndr$$

$$+\sum_{p=0}^{n}l_{n,p}(\mu+\sigma)\int_0^{p}\sum_{q=0}^{m}\int_0^{\tau}\varphi_m(f(n,\cdot);\ q+r)l_{m,q}(\nu+\tau-r)\,dndr$$

$$+\sum_{q=0}^{m}l_{m,q}(\nu+\tau)\int_0^{q}\sum_{p=0}^{n}\int_0^{\sigma}\varphi_n(f(\cdot,r);\ p+u)l_{n,p}(\mu+\delta-n)dndr$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\int_0^{\sigma}\int_0^{\pi}\varphi_{n,m}(f;\ p+n,\ q+r)\,l_{n,p}(\mu+\sigma-n)\,l_{m,q}(\nu+\tau-r)\,dndr.$$

Proof. In (2.3) we substitute $f(x)$ by $f(x,y)$, where $y$ is fixed and obtain

(2.5) $$\int_0^{\mu+\sigma}f(n,y)dn+\sum_{p=0}^{n}l_{n,p}(\mu+\sigma)\int_0^{p}f(n,y)dn$$

$$+\sum_{p=0}^{n}\int_0^{\sigma}\varphi_n(f(\cdot,y);\ p+n)l_{n,p}(\mu+\varphi-n)dn.$$

From the definition of the operator $\varphi_n(f,x)$, it is evident that

(2.6) $$\varphi_n(\varphi_m(f;\ q+\tau);p+\sigma)=\varphi_m(\varphi_n(f;\ p+\sigma);\ q+\tau)=\varphi_{n,m}(f;\ p+\sigma,\ q+\tau)$$

for all non-negative integers $n$, $m$ (See (1.6) and (1.7)). In order to complete the proof, we apply (2.3) to the function $g(y)=\int_0^{\mu+\sigma}f(u,y)du$ and from (2.5), (2.6) Theorem 3 follows.

If we differentiate (2.4) by $x$ and respectively by $y$, the following basic representation of $f(x,y)$ is obtained, which holds almost everywhere in $\square$:

(2.7) $$f(\mu+\sigma,\ \nu+\tau)=\sum_{p=0}^{n}\sum_{q=0}^{m}l'_{n,p}(\mu+\sigma)\,l'_{m,q}(\nu+\tau)\int_0^{p}\int_0^{q}f(n,r)dndr$$

$$+\sum_{p=0}^{n} l'_{n,p}(\mu+\sigma)\int_0^p \sum_{q=0}^{m} \varphi_m(f(n,\cdot);\, q+\tau)\, l_{m,q}(\nu)\, dn$$

$$+\sum_{p=0}^{n} l'_{n,p}(\mu+\sigma)\int_0^p \sum_{q=0}^{m}\int_0^r \varphi_m(f(n,\cdot);\, q+r)\, l'_{m,q}(\nu+\tau-r)\, drdn$$

$$+\sum_{q=0}^{m} l'_{m,q}(\nu+\tau)\int_0^q \sum_{p=0}^{n} \varphi_n(f(\cdot\,,r);\, p+\sigma) l_{n,p}(\mu)\, dr$$

$$+\sum_{q=0}^{m} l'_{m,q}(\nu+\tau)\int_0^q \sum_{p=0}^{n}\int_0^\sigma \varphi_n(f(\cdot\,,r);\, p+n) l'_{n,p}(\mu+\sigma-n)\, dudr$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m} \varphi_{n,m}(f;\, p+\sigma,\, q+\tau)\, l_{n,p}(\mu)\, l_{m,q}(\nu)$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\int_0^\sigma \varphi_{n,m}(f;\, p+n,\, q+\tau)\, l'_{n,p}(\mu+\sigma-n)\, l_{m,q}(\nu)\, dn$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\int_0^\tau \varphi_{n,m}(f;\, p+\sigma,\, q+r)\, l_{n,p}(\mu)\, l'_{m,q}(\nu+\tau-r)\, dr$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\int_0^\sigma\int_0^\tau \varphi_{n,m}(f;\, p+n,\, q+r)\, l'_{n,p}(\mu+\sigma-n)\, l'_{m,q}(\nu+\tau-r)\, drdn.$$

Using (1.7), the right-hand side of (2.7) can be written in the form $f(x,y)+\Phi(x,y)$, where $\Phi(x,y)$ is a continuous function and $\Phi(x,y)=0$ almost everywhere in $\square$. Now it is clear that (2.7) is true for every point of $\square$.
From (2.7) we have

$$f(\mu+\sigma,\, \nu+\tau)-P^*(f;\, x,\, y)=\varphi_{n,m}(f;\, \mu+\sigma,\, \nu+\tau) \tag{2.8}$$

$$+\sum_{p=0}^{n}\int_0^\sigma \varphi_{n,m}(f;\, p+n,\, \nu+\tau)\, l'_{n,p}(\mu+\sigma-n) dn$$

$$+\sum_{q=0}^{m}\int_0^\tau \varphi_{n,m}(f;\, \mu+\sigma,\, q+r)\, l'_{m,q}(\nu+\tau-r)\, dr$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\int_0^\sigma\int_0^\tau \varphi_{n,m}(f;\, p+n,\, q+r)\, l'_{n,p}(\mu+\sigma-n)\, l'_{m,q}(\nu+\tau-r)\, dudr,$$

where $P^*(f;\, x,\, y)\in Q_{n-1,\, m-1}(\square)$.

Theorem 4. *If $f\in L(\square)$, then there is a quasipolynomial $R(f;\, x,\, y)\in Q_{n,m}(\square)$ satisfying the following conditions:*

(i) $\int_i^{i+1}\int_j^{i+1} [f(x,y)-R(f;\, x,\, y)]\, dxdy=0$ for $i=0,\, 1,\dots,n$; $j=0,\, 1,\dots,m$;

(ii) $\int_j^{i+1} [f(x,y)-R(f;\, x,\, y)]\, dy=0$ for $j=0,\, 1,\dots,m$;

(iii) $\int_i^{i+1} [f(x,y)-R(f;\, x,\, y)]\, dx=0$ for $i=0,\, 1,\dots,n$;

(iv) $$\Delta^{n,m}_{\zeta,\eta} R(f;\ x,\ y) = -\int_0^1\int_0^1 \Delta^{n,m}_{1,1} f(n,\ r)\ dndr\,\zeta^n\eta^m + \int_0^1 \Delta^{n,m}_{1,\eta} f(n,\ y)\,dn\,\eta^m$$

$$+\int_0^1 \Delta^{n,m}_{\zeta,1} f(x,\ r)dr\,\zeta^n,\quad 0\leqq\zeta,\ \eta\leqq 1.$$

Proof. Let $F(x,\ y) = \int_0^x\int_0^y f(u,\ v)dudv$; $F_x = \int_0^y f(x,\ y)\,dv$ and $F_y(x,\ y) = \int_0^x f(u,\ y)\,du$. Then we set

(2.9) $$R\ (f;\ x,\ y) = \sum_{p=0}^{n+1} F_y(p,\ y)l'_{n+1,p}(x) + \Sigma\, F_x(x,\ q)\, l'_{m+1,q}(y)$$

$$-\sum_{p=0}^{n+1}\sum_{q=0}^{m+1} F(p,\ q)\, l'_{n+1,p}(x)\, l'_{m+1,q}(y).$$

We start to prove (ii).

$$\int_j^{j+1} [f(x,\ y) - R(f;\ x,\ y)]dy = \int_j^{j+1} f(x,\ y)\,dy - \sum_{p=0}^{n+1}\int_j^{j+1} F_y(p,\ y)\,dy\, l'_{n+1,p}(x)$$

$$-\sum_{q=0}^{n+1} F_x(x,\ q)\,[l_{m+1,q}(j+1) - l_{m+1,q}(j)]$$

$$+\sum_{p=0}^{n+1}\sum_{q=0}^{m+1} F(p,\ q)\, l'_{n+1,p}(x)\,[l_{m+1,q}(j+1) - l_{m+1,q}(j)]$$

$$= F_x(x,\ j+1) - F_x(x,\ j) - \sum_{p=0}^{n+1} [F(p,\ j+1) - F(p,\ j)]\, l_{n+1,p}(x)$$

$$-F_x(x,\ j+1) + F_x(x,\ j) + \sum_{p=0}^{n+1} [F(p,\ j+1) - F(p,\ j)]\, l_{n+1,p}(x) = 0.$$

By analogy with (ii), we get (iii). It is easy to see that (i) follows from (ii) and (iii). In order to complete the proof of theorem 4, we have to prove (iv).
From (2.9) we get

(2.10) $$\Delta^{n,m}_{\zeta,\eta} R(f;\ x,\ y) = \sum_{i=0}^{n+1}\int_0^i \Delta^m_\eta f(n,\ y)\,dn\binom{n+1}{i}(-1)^{n+1+i}\zeta^n$$

$$+\sum_{j=0}^{m+1}\int_0^j \Delta^n_\zeta f(x,\ r)\,dr\binom{m+1}{j}(-1)^{m+1+j}\eta^m$$

$$-\sum_{i=0}^{n+1}\sum_{j=0}^{m+1}\int_0^i\int_0^j f(n,\ r)\,dndr\binom{n+1}{i}\binom{m+1}{j}(-1)^{n+m+i+j}\zeta^n\eta^m,$$

(2.11) $$\sum_{i=0}^{n+1}\int_0^i \Delta^m_\eta f(n,\ y)\,dn\binom{n+1}{i}(-1)^{n+1+i}$$

$$= \sum_{i=0}^{n+1}\int_0^i \Delta^m_\eta f(n,\ y)dn\left[\binom{n}{i-1} + \binom{n}{i}\right](-1)^{n+1+i}$$

$$= \sum_{i=0}^{n} \Big[ \int_0^{i+1} \Delta_\eta^m f(\eta, y)\binom{n}{i}(-1)^{n+i} d\eta - \int_0^{i} \Delta_\eta^m f(\eta, y)\binom{n}{i}(-1)^{n+i} d\eta \Big]$$

$$= \sum_{i=0}^{n} (-1)^{n+1}\binom{m}{i} \int_i^{i+1} \Delta_\eta^m f(\eta, y)\, d\eta = \sum_{i=0}^{n} \int_0^1 (-1)^{n+i}\binom{n}{i} f(\eta+i, y) d\eta$$

$$= \int_0^1 \Delta_{1,\eta}^{n,m} f(\eta, y)\, du.$$

From (2.11) it is easy to see that

$$\sum_{j=0}^{m} \int_0^{j} \binom{m+1}{j}(-1)^{m+1+j} \Delta_\zeta^n f(x, r)dr = \int_0^1 \Delta_{\zeta,1}^{n,m} f(x, y)\, dr. \tag{2.12}$$

Applying (2.11) and (2.12), we obtain (iv).

**3. Main result.** The aim of this section is to prove the following theorem of Whitney's type $R^2$.

Theorem 5. *If $f \in L(\square)$, then there is a quasipolynomial $P \in Q_{n-1,m-1}(\square)$ such that*

$$\| f(x, y) - P(f;\ x, y) \|_\square \leq 49\, \omega_{n,m}(f;\ \square). \tag{3.1}$$

Proof. By $\square_i$, $i=1, 2, 3, 4$, we denote the following four subsets of $\square$:

$$\square_1 := \{(x, y) \in \square:\ 0 \leq x \leq n,\ 0 \leq y \leq m\};$$

$$\square_2 := \{(x, y) \in \square:\ 1 \leq x \leq n+1,\ 0 \leq y \leq m\};$$

$$\square_3 := \{(x, y) \in \square:\ 0 \leq x \leq n,\ 1 \leq y \leq m+1\};$$

$$\square_4 := \{(x, y) \in \square:\ 1 \leq x \leq n+1,\ 1 \leq y \leq m+1\}.$$

Obviously,

$$\bigcup_{i=1}^{4} \square_i = \square. \tag{3.2}$$

Let us mention that if we substitute $f$ by $\tilde f = f(x, y) - R(f;\ x, y)$ ($R(f;\ x, y)$ is as in Theorem 4), then from (2.7) and (2.8) we get

$$P^*_{\square_i}(g;\ x, y) = P^*_{\square_i}(\tilde f;\ x, y) = 0, \quad \text{for } i=1, 2, 3, 4, \tag{3.3}$$

where $g(x, y) = \tilde f(n+1-x,\ m+1-y)$.

We call the assertion (3.3) a condition for symmetry.

From (2.8), (3.3) and Theorem 4 we have

$$\begin{aligned} \tilde f(x, y) &= f(x, y) - R(f;\ x, y) = \varphi_{n,m}(\tilde f;\ p+\sigma,\ q+\tau) \\ &+ \sum_{p=0}^{n} \int_0^{\sigma} \varphi_{n,m}(\tilde f;\ p+u,\ q+\tau)\, l'_{n,p}(\mu+\sigma-\eta)\, \partial\eta \\ &+ \sum_{q=0}^{m} \int_0^{\tau} \varphi_{n,m}(\tilde f;\ p+\sigma,\ q+r)\, l'_{m,q}(\nu+\tau-r)\, dr \\ &+ \sum_{p=0}^{n} \sum_{q=0}^{m} \int_0^{\sigma} \int_0^{\tau} \varphi_{n,m}(\tilde f;\ p+\eta,\ q+r)\, l'_{n,p}(\mu+\sigma-\eta)\, l'_{m,q}(\nu+\tau-r) d\eta dr, \end{aligned} \tag{3.4}$$

Applying Lemma 1, Lemma 2 and (1.7), we obtain

$$(3.5)\qquad |f(x, y)-R(f; x, y)| \leq \omega_{n,m}(\tilde{f}; \square)/\binom{n}{\mu}\binom{m}{\nu}$$

$$+\sum_{p=0}^{n}\frac{\omega_{n,m}(\tilde{f}; \square)}{\binom{n}{p}\binom{m}{\nu}}\max_{\mu\leq t\leq\mu+1}|l_{n,p}(t)|+\sum_{q=0}^{m}\frac{\omega_{n,m}(\tilde{f}; \square)}{\binom{n}{\mu}\binom{m}{q}}\max_{\nu\leq t\leq\nu+1}|l_{m,q}(t)|$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{m}\frac{\omega_{n,m}(\tilde{f}; \square)}{\binom{n}{p}\binom{m}{q}}\max_{\mu\leq t\leq\mu+1}|l_{n,p}(t)|\max_{\nu\leq t\leq\nu+1}|l_{m,q}(t)|$$

$$\leq \omega_{n,m}(\tilde{f}; \square)\frac{1+1+\sigma_\mu+\sigma_{\mu+1}+1+\sigma_\nu+\sigma_{\nu+1}+(1+\sigma_\mu+\sigma_{\mu+1})(1+\sigma_\nu+\sigma_{\nu+1})}{\binom{n}{\mu}\binom{m}{\nu}}.$$

From (iv) of Theorem 4, it is obvious that

$$\omega_{n,m}(\tilde{f}; \square)\leq 4\omega_{n,m}(f; \square).$$

Therefore,

$$(3.6)\quad \|f(x, y)-R(f; x, y)\|_\square \leq 4\omega_{n,m}(f; \square)(1+2+2+4)\leq 36\omega_{n,m}(f; \square).$$

Now using (2.11), we may represent $R(f; x, y)\in Q_{n,m}(\square)$ in the form

$$(3.7)\quad R(f; x, y)=P(f; x, y)+\Big[\int_0^1 \Delta_1^n f(n, y)\,du-S_{m-1}(y)\Big]\frac{\prod_{j=1}^{n}(x-j)}{n!}$$

$$+\Big[\int_0^1 \Delta_1^m f(x, r)dr-S_{n-1}(x)\Big]\frac{\prod_{j=1}^{m}(y-j)}{m!}-\int_0^1\int_0^1 \Delta_{1,1}^{n,m} f(n, r)\,du\,dr$$

$$\times\frac{\prod_{i=1}^{n}(x-i)\prod_{i=1}^{m}(y-j)}{n!\,m!},$$

where $S_{n-1}(x)$, $S'_{m-1}(y)$ are the polynomials from Whitney's Theorem 2 corresponding to the functions $\int_0^1 \Delta_1^m f(x, r)\,dr$, $\int_0^1 \Delta_1^n f(n, y)dn$. It is evident that $P(f; x, y)\in Q_{n-1,m-1}(\square)$.

From (3.6), (3.7) and Theorem 2 we have

$$\|f(x, y)-P(f; x, y)\|_\square\leq 36\,\omega_{n,m}(f; \square)$$

$$+\Big\|\int_0^1 \Delta_1^m f(x, r)\,dr-S_{n-1}(x)\Big\|_\square+\Big\|\int_0^1 \Delta_1^n f(n, y)dn-S_{m-1}(y)\Big\|_\square$$

$$+\Big\|\int_0^1\int_0^1 \Delta_{1,1}^{n,m} f(u, r)\,dn\,dr\Big\|_\square\leq\omega_{n,m}(f; \square)(36+6+6+1)=49\,\omega_{n,m}(f; \square).$$

**4. Remarks and generalizations.** Theorem 5 may be formulated and proved for integral $L_p$-norm ($p\geq 1$). Whitney's theorem in $R^1$ for integral norm has been proved in [6].

Using induction, we may generalize Theorem 5 in $R^N$, where $N=3, 4, 5, \ldots$.

The essential difference between Theorem 5 and the work of J. Brudnyi [7] is that the constant $C$ is independent of $n$ and $m$. The constant $C=49$ in Theorem 5 obviously is not the best possible.

## REFERENCES


1. H. Whitney. On functions with bounded $n$-th differences. *J. Math. Pures Appl.*, **36**, 1957, 67-94.
2. H. Whitney. On bounded functions with bounded $n$-th differences. *Proc. Amer. Math. Soc.*, **10**, 1959, 480-481.
3. K. G. Ivanov. New estimates of quadrature formulae, formulae of numerical differentiation and interpolation. *Anal. Math.*, **6**, 1980, 281-303.
4. Bl. Sendov. The constants of H. Whitney are bounded. *C. R. Acad. Bulg. Sci.*, **38**, 1985, 10, 1299-1302.
5. Bl. Sendov. On the theorem and constants of H. Whitney. *Constr. Approx.*, 1987.
6. Bl. Sendov, M. D. Takev. The theorem of Whitney for integral norm. *C. R. Acad. Bulg. Sci.*, **39**, 1986, 10, 35-38.
7. Ю. А. Брудный. Приближение функций $n$ переменных квазимногочленами. *Изв. АН СССР, Сер. мат.*, **34**, 1970, 564-583.


*Institute of Mathematics*
*Bulgarian Academy of Sciences*
*P. O. Box 373, Sofia, Bulgaria*

*Received 9. 12. 1986*
*Revised 14.04.1987*